Mechanical Formulae

Term	Description	Unit
d	Diameter	m
F	Force	N
g	Acceleration due to gravity	ms^{-2}
J	Total inertia	kgm^2
J_L	Load inertia	kgm^2
J_M	Motor inertia	kgm^2
m	Mass	kg
M	Motor torque	Nm
M_a	Accelerating torque	Nm
M_L	Load torque	Nm
n	Rotational frequency	rpm*
$n1$	- input	rpm*
$n2$	- output	rpm*
Δn	Change of rotational frequency	rpm*
p	Pitch	m
P	Motor power	kW
P_a	Accelerating power	kW
P_L	Load power absorbed	kW
r	Radius	m
s	Distance	m
t	Acceleration time	s
Δt	Acceleration period	s
v	Linear velocity	m/min*
Δv	Change of linear velocity	m/min*
V	Traction capacity	M^3s^{-1}
W	Energy	J (Joule)
η	Efficiency	-
μ	Coefficient of friction	-

Note: For practical convenience, some of the units in the formulae following are not S1 units; for example, rotational frequency is commonly measured in revolutions per minute, although the S1 unit is revolutions per second. In these Servo Formulae, the terms used are as tabulated above.Those which are in non-S1 units are marked *.

Linear Motion

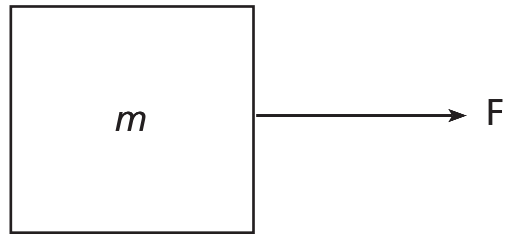


Fig. A

Consider a body mass m acted upon by a single force F, Fig A. The body accelerates in the direction in which the force is acting, at a rate given by:

$$A = F/m$$

After a time t has elapsed, the body has achieved a velocity v, where:

$$v = u + at$$

(u is the initial velocity, before the force F was applied. If the body was initially at rest, u is zero)

The distance, s, travelled by the body during time t is

$$s = ut + at^2/2$$

Distance and velocity are related by the following equation, derived from the two previous ones:

$$v^2 - u^2 = 2as$$

The work done by the force in accelerating the body is the product of force and distance:

$$W = Fs$$

The kinetic energy of the body, ie the energy which it possesses by virtue of its motion, is the product of its mass and the square of its velocity:

$$E_k = mv^2/2$$

Furthermore, since energy is conserved, the work done by the force is equal to the change in the body's kinetic energy (neglecting losses):

$$W = m(v^2 - u^2)/2$$

Power is the rate at which work is done, therefore it is the product of force and velocity:

$$P = Fv$$

Rotational or Angular Motion

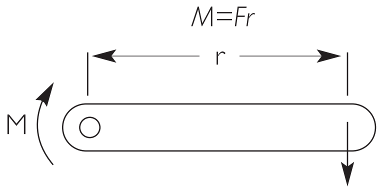


Fig A.11
The concept
of torque

A force acting perpendicular to a pivoted lever, Fig A.11, causes a turning effect or torque at the fulcrum. The torque is the product of the force and the radius at which it is applied.

$$M = Fr$$

If a torque is applied to a body which is free to rotate, as in Fig A.12, an acceleration results in a way which is analogous to the example of linear motion above. Indeed a similarity will be noticed between the equations of motion.

Any body which is capable of rotating possesses a property known as Moment of Inertia which tends to resist acceleration in the same way as does the mass of a body in linear motion. The moment of inertia is related not only to the mass of the body, but also to the distribution of that mass with respect to radius.

The moment of inertia of a solid cylinder of radius r is given by:

$$J = mr^2/2$$

By comparison, the moment of inertia of a hollow cylinder, of inner and outer radii respectively, is as follows:

$$J = m(r_o^2 - r_i^2)/2$$

It can be seen that, for a given outer radius, the moment of inertia of a hollow cylinder is greater than that of a solid cylinder of the same mass. In Fig A.12, a body having a moment of inertia J is acted upon by a torque M. Its angular acceleration is:

$$\alpha = M/J$$

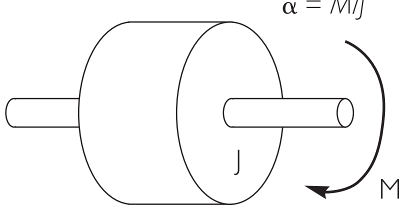


Fig A.12
The Action of
torque on a body

After a time t has elapsed, the angular velocity, ω (rate of change of angle) is given by:

$$\omega = \omega_o + \alpha t$$

(wo is the initial angular velocity, before the torque M was applied. If the body was initially at rest, ω_o is zero)

The angle, g, through which the body rotates in time t is:

$$\gamma = \omega_o t + \alpha t^2/2$$

Angle and angular velocity are related by the following equation:

$$\omega^2 - \omega_o^2 = 2\alpha\gamma$$

The work done in accelerating the body is the product of torque and angle of rotation:

$$W = M\gamma$$

The kinetic energy of the body is the product of its moment of inertia and the square of its angular velocity:

$$E_k = J\omega^2/2$$

Since energy is conserved, the work done is equal to the change in kinetic energy (neglecting losses):

$$W = J(\omega^2 - \omega_o^2)/2$$

Power is the product of torque and angular velocity, i.e. the rate at which work is being done:

$$P = M\omega$$

Relationship between linear and angular motion

Consider a body of mass m moving in a circle of radius r with an angular velocity ω, Fig A.13.

When the body has rotated through an angle γ, it has covered a distance s along circumference of the circle, where:

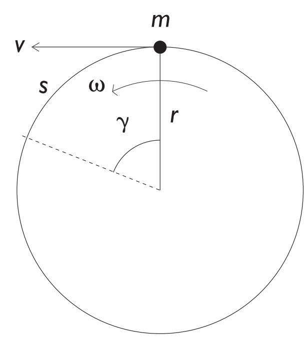


Fig A.13
Relationship
between linear and
angular motion

Similarly, the tangential velocity or peripheral speed v, being the quotient of distance and time, is given by:

$$v = s/t = \gamma r/t$$

Angular velocity w is the quotient of angle and time;

$$w = \gamma/t$$

Therefore

$$v = wr$$

Similarly, for acceleration:

$$a = v/t = wr/t$$

$$\alpha = w$$

Therefore

$$a = \alpha r$$

The moment of inertia is given by

$$J = mr^2$$

The Effect of Gearing

When calculating the torque required to accelerate or decelerate the moving parts of a machine, it is necessary to take into account any gearing which introduces a ratio between the speeds of different parts. It is unusual to calculate the moment of inertia transferred to the motor shaft, since this figure may be added arithmetically to the motor inertia to arrive at a figure for the total inertia of the system. Fig A.14 illustrates a motor, having a moment of inertia J_1, driving a load with inertia J_2, via a gearbox.

If the gearbox has a ratio k, then the relationship between input and output angular velocities is as follows:

$$\omega_1 = k\omega_2$$

Neglecting losses, the input and output torques are related thus:

$$M_1 = M_2/k$$

The load inertia reflected back through the gearbox to the motor shaft is reduced by a factor equal to the square of the gear ratio. Therefore the total inertia which the motor has to overcome is given by:

$$J = J_1 + J_2 / k_2$$

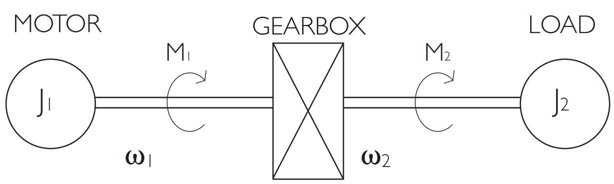
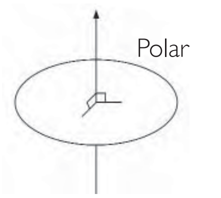
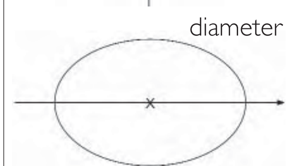
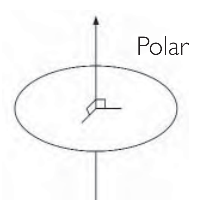
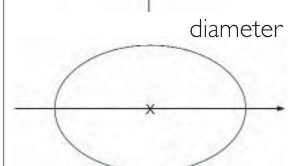
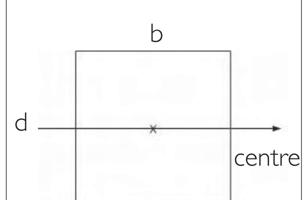
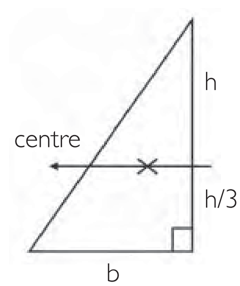
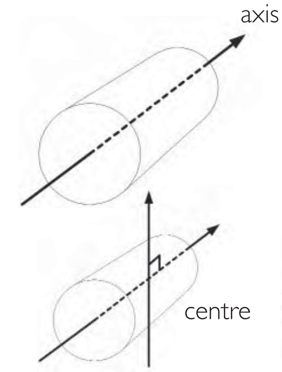
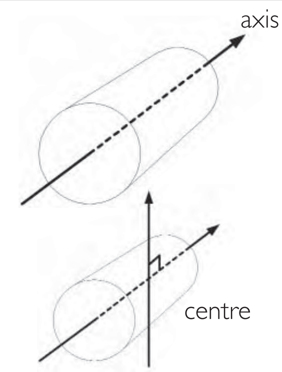
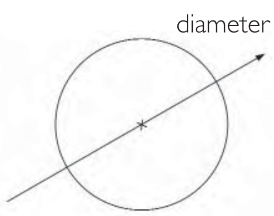
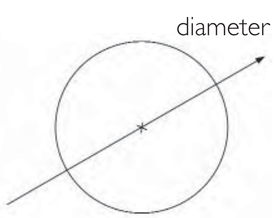


Fig A.14 The effect of gearing between motor and load

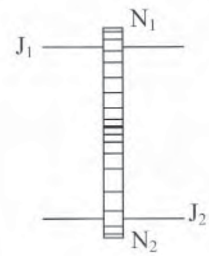
INERTIA

Body	Axis	Moment of Inertia(J)
Uniform rod (length l)		$\frac{ml^2}{12}$
Uniform hoop (radius r, diameter d)	 	mr^2 $\frac{1}{2}mr^2$
Uniform thin disk (radius r, diameter d)	 	$\frac{1}{2}mr^2$ $\frac{1}{4}mr^2$
Rectangular Plate		$\frac{md^2}{12}$
Triangular Plate		$\frac{mh^2}{18}$

Symbols & Formulae

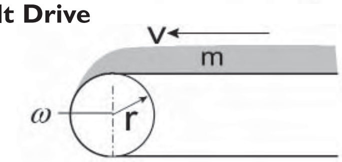
Body	Axis	Moment of Inertia(J)
Solid circular cylinder (radius r)		$\frac{1}{2}mr^2$ $m\left(\frac{I^2}{12}+\frac{r^2}{4}\right)$
Cylindical shell (no ends)		mr^2 $\frac{m}{12}\left(12r^2+5I^2\right)$
Uniform Solid Sphere (radius r)		$\frac{2mr^2}{5}$
Uniform (radius r)		$\frac{2mr^2}{3}$

Gear Reduction



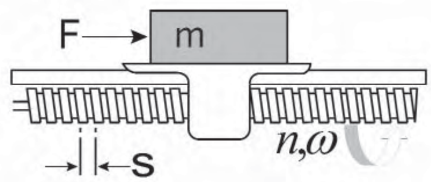
$$J_1 = \left(\frac{n_1}{n_2}\right)^2 .J_2$$

Belt Drive



$$J = m.r^2$$
$$J = m(v/\omega)^2$$

Screw Drive



$$J = m\left(\frac{s}{2\pi}\right)^2$$

Inertia Matching

1. For extremely fast acceleration, use 1:1 inertia match.
2. For minimum peak power, use reflected load inertia 2.5 times motor inertia.
3. The general "rule of thumb" is to avoid inertia higher than 5 times motor inertia, however "Motion Made Easy" space-state control enables mismatches to 10 times motor inertia.

CONTROL TECHNIQUES

Servo Formulae

Motor Torque Constant $K_T = \frac{M}{I}(Nm/A)$

Motor Input Volts $V = IRm + K_E W + L\frac{di}{dt}$

Motor Regulation $= \frac{R}{K_E K_T}$

Developed Torque $M = M_L + \omega F_I + \alpha(Jm + J_L)$

RMS Torque, $M_{RMS} = \sqrt{\frac{M_1^2 t_1 + M_2^2 t_2 + M_3^2 t_3}{(t_1+t_2+t_3)}}$

	Linear	Rotary
Distance, Angle(m, rads)	$s = vt$	$\theta = \omega t$
Velocity, (m/s, rads/second)	$v = s/t$	$\omega = 2\pi n$
Acceleration, (m/s^2, $rads/s^2$)	$a = v/t$	$a = \omega/t$
Force (N)	$F = m\,a$	
Torque (Nm)	$M = F\,r$	$M = ja$
Power (W)	$P = F\,v$	$P = M\,w$
Kinetic Energy (J)	$W = \frac{1}{2}mv^2$	$w = \frac{1}{2}j\omega^2$

Motion Equations

Velocity:

$$v = u + at$$

$$v^2 = u^2 + 2\,a\,s$$

Distance:

$$s = ut + \frac{1}{2}at^2$$

Area, Volume and Arc Length

Sphere: $V = \frac{4}{3}\pi r^3, S = 4\pi r^2$

Circular Cone: $V = \frac{1}{3}r^2 h, S = \pi r l$

(h= vertical height,
l = slant height,
r = base radius & $l^2 = h^2 + r^2$)

Circular Cylinder: $V = \pi r^2 h, S = 2\pi r l$

Pyramid: $V = \frac{1}{3}$ x (base) x (perpendicular height)

Circle: $A = \frac{1}{2}r^2\theta, I = r\theta$

Frustrum of Cone: $A = \pi\frac{h}{3}\left(R^2 + Rr = r^2\right)$

Triangle:

$$A = \frac{1}{2}ab\sin C$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$2s = a + b + c$$

Eclipse: $A = \pi ab$

Spherical cap:

$$V = \frac{1}{3}\pi h^2(3r - h)$$

$$A = 2\pi + h$$